Which Parameters to Simulate?

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Outline

Which parameters to simulate?

Buckingham's π-theorem

Normalizing equations

Application to plasmas

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Which parameters to simulate?

Typical case: My problem depends on too many parameters! How do I choose?

Example: Obtain drag on sphere in a viscous flow.

$$F_d = f(D, U, \rho, \mu)$$

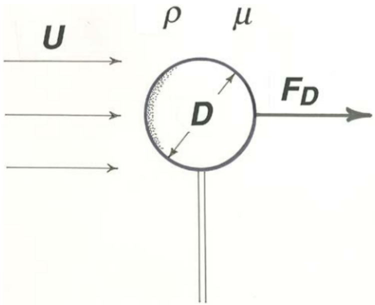

Figure: Viscous flow past sphere.
https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-090-introduction-to-fluid-motions-sediment-transport-and-current-generated-sedimentary-structures-fall-2006/course-textbook/ch2.pdf

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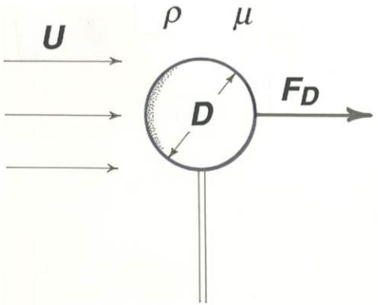
Sweeping 4 variables over 10 values $= 10^4$ simulations!



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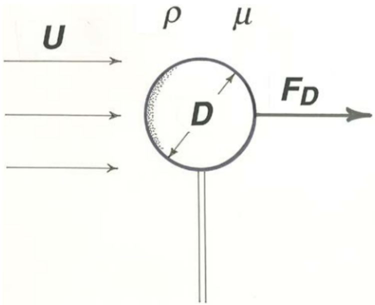
Useful techniques:

- Buckingham's π-theorem
- Normalize governing equations
- Physical arguments (e.g., $n_i \approx n_e$)
- Look for common quantities



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Buckingham's π-theorem: Flow past sphere

5 parameters (1 out, 4 in):

- Drag F_d $[\mathrm{kg\,m\,s^{-2}}]$
- Diameter D $[\mathrm{m}]$
- Flow velocity U $[\mathrm{m\,s^{-1}}]$
- Mass density ρ $[\mathrm{kg\,m^{-3}}]$
- Viscosity μ $[\mathrm{kg\,m^{-1}\,s^{-1}}]$

3 base units (m, kg, s)
$5 - 3 = 2$ indep. dim.less groups

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Input:

1. Eliminate kg: $\frac{\mu}{\rho}$ $[\mathrm{m^2\,s^{-1}}]$
2. Eliminate s: $\frac{\mu}{\rho U}$ $[\mathrm{m}]$
3. Eliminate m: $\pi_1 = \frac{\mu}{\rho U D} = \mathrm{Re}$ $[1]$

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Output :

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There is a relation $\mathcal{F}(\pi_1, \pi_2) = 0 \quad \Rightarrow \quad \frac{F_d}{\rho U^2 D^2} = f(\mathrm{Re})$ – Sweep 1 variable!

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3. Eliminate m: $\pi_1 = \frac{\mu}{\rho U D} = \mathrm{Re}$ $[1]$

Output (alternative):

1. Eliminate kg: $\frac{F_d}{\mu}$ $[\mathrm{m^2\,s^{-1}}]$
2. Eliminate s: $\frac{F_d}{\mu U}$ $[\mathrm{m}]$
3. Eliminate m: $\pi_2' = \frac{F_d}{\mu U D} = \frac{\pi_2}{\mathrm{Re}}$ $[1]$

There is a relation $\mathcal{F}(\pi_1, \pi_2') = 0 \quad \Rightarrow \quad \frac{F_d}{\mu U D} = f(\mathrm{Re})$ – Sweep 1 variable! Still works!

Buckingham's π-theorem: A simple pendulum

Assume period is a function $T = f(g, l, m)$

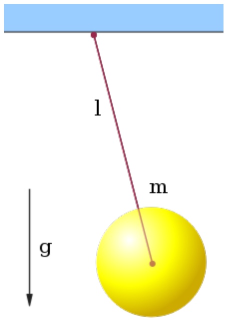


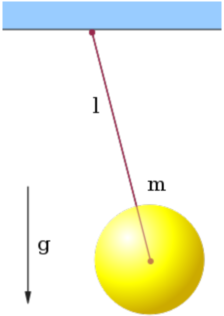
Figure: From Wikipedia

Buckingham's π-theorem: A simple pendulum

Assume period is a function $T = f(g, l, m)$

4 parameters − 3 base units (kilogram, second, meter) = 1 group:

$$\pi = \frac{gT^2}{l} \quad \Rightarrow \quad \mathcal{F}\left(\frac{gT^2}{l}\right) = 0$$



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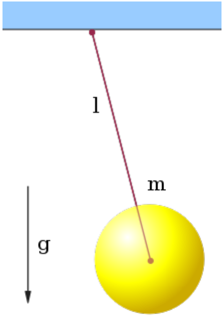

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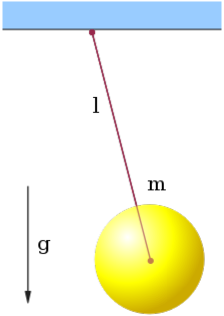
Can also use linear algebra:

$$\pi = T^{\alpha_1} g^{\alpha_2} l^{\alpha_3} m^{\alpha_4} \quad [\mathrm{kg}^{\beta_1}\ \mathrm{m}^{\beta_2}\ \mathrm{s}^{\beta_3}]$$

$$\beta_1 = \beta_2 = \beta_3 = 0$$



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Outline

Normalize governing equations: Flow past sphere

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u}$$

3 base units (m, kg, s) $\Rightarrow$ use 3 normalizing constants (D, U, ρ):

$$\mathbf{x}' = \frac{\mathbf{x}}{D}, \quad \mathbf{u}' = \frac{\mathbf{u}}{U}, \quad p' = \frac{p}{\rho U^2}, \quad t' = \frac{tD}{U}$$

$$\frac{D\mathbf{u}'}{Dt'} = -\nabla' p' + \underbrace{\frac{\mu}{\rho U R}}_{\mathrm{Re}} \nabla'^2\mathbf{u}'$$

Truly only one input parameter: Re.

Also only one way to non-dimensionalize F_d using D, U, ρ (try it!)

Normalize governing equations: Flow past sphere

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u}$$

3 base units (m, kg, s) $\Rightarrow$ use 4 normalizing constants (D, U, ρ, T):

$$\mathbf{x}' = \frac{\mathbf{x}}{D}, \quad \mathbf{u}' = \frac{\mathbf{u}}{U}, \quad p' = \frac{p}{\rho U^2}, \quad t' = \frac{D}{T}$$

$$\frac{D\mathbf{u}'}{Dt'} = -\underbrace{\frac{UT}{R}}_{\text{param. 1}}\nabla' p' + \underbrace{\frac{\mu T}{\rho R^2}}_{\text{param. 2}}\nabla'^2\mathbf{u}'$$

Coefficient split in two $\Rightarrow$ Did not work!

F_d can be non-dimensionalized in multiple ways using D, U, ρ, and T (try it!)

Normalize governing equations: Flow past sphere

Normalizing constants are actually new units. Another example of "too many" units:

Heaviside-Lorentz units (cm, g, s)

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{1}{c}\left(\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)
\end{aligned}
$$

One constant: c

SI units (m, kg, s, A)

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\varepsilon_0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)
\end{aligned}
$$

Two constants: ε_0, μ_0.

Outline

Example: finite length cylinder

Find current due to attracted species s for finite length cylinder:
$I_s = f(V, q_s, m_s, n_s, T_s, r, l, \varepsilon_0)$ – 9 parameters

1. Look for common quantities to reuse in OML theory:

$$I_s = C \underbrace{A q_s n_s \sqrt{\frac{kT_s}{2\pi m_s}}}_{I_{\mathrm{th,s}}} \Big(1 - \underbrace{\frac{q_s V}{kT_s}}_{\eta_s}\Big)^{0.5}$$

2. Buckingham's π-theorem:

$$\mathcal{F}\left(\frac{I_s}{I_{\mathrm{th},s}}, \eta_s, \frac{l}{\lambda_D}, \frac{r}{\lambda_D}, n\lambda_D^3\right) = 0$$

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2. Buckingham's π-theorem:

$$\mathcal{F}\left(\frac{I_s}{I_{\mathrm{th},s}}, \eta_s, \frac{l}{\lambda_D}, \underbrace{\cancel{\frac{r}{\lambda_D}}}_{\to 0}, \underbrace{\cancel{n\lambda_D^3}}_{\to\infty}\right) = 0 \quad \Rightarrow \quad I_s = I_{\mathrm{th},s} f\left(\eta_s, \frac{l}{\lambda_D}\right)$$

3. Use physical arguments

Marholm and Marchand, DOI: 10.1103/PhysRevResearch.2.023016